Analytic Spec

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A Banach algebra is a $\mathbb{C}$ algebra A that is also a Banach space and the norm has the following compatability condition with the algebra structure

$$\|xy\| \leq \|x\|\|y\|$$

we call a Banach algebra unital if the algebra is unital. A homomorphism of Banach algebras is an algebra homomorphism that is also bounded. From now on we fix a commutative unital Banach algebra A; then we define its spectrum

$$\sigma(A) \quad := \quad \{\text{non-zero homomorphisms from } A \text{ into } \mathbb{C}\}$$

Given an element $T \in A$ then we get a function $\hat{T} = ev_T : \sigma(A) \to \mathbb{C}$ by

$$\hat{T}(h) = h(T)$$

it is through this that the connection with the spectrum is made.

let $\mathcal{H}$ be a seperable Hilbert space (over $\mathbb{C}$), and $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators from $\mathcal{H}$ to itself. Notice that for a fixed $T \in \mathcal{L}(\mathcal{H})$ the (Banach) subalgebra it generates will always be commutative and by definition unital, denote this algebra $\langle T \rangle$. Now by [Fol16, Prop 1.15] (ultimately the spectral theorem) we know that actually

$$\hat{T} = ev_T : \sigma\langle T \rangle \to \sigma(T)$$

is a homeomorphism.

We now connect this with the world of algebraic geometry

Theorem. *[Fol16, 1.12] The map*

$$h \mapsto ker(h)$$

gives a bijection

$$\sigma(A) \to \{\textit{maximal ideals of } A\}$$

Applying this to the case above we see that $\langle T \rangle = \mathbb{C}[T]$ (this equality was pointed out to me by Matt Emerton in a talk) hence

$$\{\text{ max ideals of } \mathbb{C}[T]\} \xleftrightarrow{ker} \sigma(\mathbb{C}[T]) \xleftrightarrow{ev} \sigma(T)$$

We remark that for a finite dimensional $\mathcal{H}$ then if $m(T)$ is the minimal polynomial of T then $\mathbb{C}[T] \cong \mathbb{C}[x]/(m(x))$.

$$\text{maxSpec } \mathbb{C}[x]/(m_T(x)) \leftrightarrow \text{maxSpec } \mathbb{C}[T] = \{\text{ max ideals of } \mathbb{C}[T]\} \xleftrightarrow{ker} \sigma(\mathbb{C}[T]) \xleftrightarrow{ev} \sigma(T)$$

If we replace the minimal polynomial with the characteristic polynomial then the powers are somehow preserved in the prime ideals, this is something Jack mentioned and is elaborated a pon in

`https://golem.ph.utexas.edu/category/2011/10/spectra_of_operators_and_rings.html#:~:text=So%2C%20s`

Another remark is that if we have a family of operators that commute, for instance a family indexed by a group (some representation) then they have the same spectrum, we can take the algebra generated by all of them and applying a generalisation of the above get something very similar.

References

[Fol16] Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Chapman and Hall/CRC, 0 edition, February 2016.